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The stability of free-surface flows with viscosity stratification

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The hydrodynamic stability is examined of liquids in which the viscosity varies with distance below a free surface. It is assumed that the viscosity becomes indefinitely large with distance from the surface, and that there is an 'effective depth' within which most of the motion occurs; but, otherwise, the viscosity distribution is arbitrary. The particular cases of wave-generation by a concurrent air flow at a horizontal liquid surface, and the stability of inclined flow under gravity are treated. It is shown that instabilities may occur which are similar to those known for thin uniform liquid films.

1. Introduction

The hydrodynamic stability of a uniform liquid film, flowing under gravity down an inclined plane, has been examined by Benjamin (1957) and Yih (1954, 1963). Also, Craik (1966) has investigated the stability of a thin uniform liquid film on a horizontal boundary, which is exposed to a concurrent air flow. In both cases, instability may exist at rather small liquid Reynolds numbers, and the unstable disturbances have wavelengths which are large compared with the film thickness.

The present work is closely related to these investigations, but differs from them in that it concerns liquids for which the viscosity varies with distance from the free surface. The liquid density, however, is assumed to be constant. The depth of the liquid may be taken as infinite, but the viscosity is assumed to increase indefinitely with depth. This situation may be regarded as a model of a melting surface.

Although the fluid is not enclosed by a rigid lower boundary, it is generally possible to define a length-scale h which, for hydrodynamic purposes, is a measure of the 'effective thickness' of the liquid layer. Such a length-scale may be specified in terms of the distribution of viscosity with depth. If the viscosity becomes very large at comparatively small depths—as is usually the case at melting surfaces—this length-scale is small, and the liquid might be expected to behave to some extent like a thin film on a rigid boundary. The present work was undertaken to discover whether the instabilities which occur in uniform thin films may also take place in this situation. Accordingly, two specific problems are examined, which respectively concern horizontal flows under the action of an air stream, and flows under gravity with a free surface inclined at an angle to the horizontal.

For the primary flow, a Reynolds number R ($\equiv \rho V h/\mu_0$) may be defined in terms of the velocity V and viscosity μ_0 at the liquid surface, the liquid density ρ and the 'effective thickness' h of the liquid. Also, if the flow experiences a small two-dimensional periodic disturbance of wave-number k, an appropriate dimensionless wave-number is $\alpha \equiv kh$. As in the previous work on uniform films, approximate solutions to the stability problem may be obtained when α^2 and αR are small. However, the variation of viscosity with depth now makes the analysis more complicated.

Here it is assumed that the viscosity of each element of the liquid remains constant throughout its motion: that is,

$$D\mu/Dt' = 0, \tag{1.1}$$

where $\mu(\mathbf{x}', t')$ is the viscosity at position $\mathbf{x}'$ and time t', and D/Dt' denotes the material time derivative. This equation is exact provided the viscosity distribution is not subject to diffusion. However, in practice, variations in viscosity are normally due to variations in temperature; and diffusion of heat may produce a corresponding diffusion of viscosity. In such cases, equation (1.1) is an approximation which is likely to be valid only when the thermal diffusivity κ of the liquid is sufficiently small. More precisely, when α^2 is small, (1.1) may be a good approximation provided

$$\alpha R Pr \gg 1,$$

where $Pr \equiv \mu_0/\rho\kappa$ is the Prandtl number. Since αR will be taken as $O(1)$ or less in the present work, it is clearly necessary that the Prandtl number of the material in question should be large. (This condition appears to be well satisfied, for example, by molten metals and glass.) Restrictions similar to that above exist for other diffusive agents, such as molecular diffusion.

A further qualification concerning (1.1) must be made. If a small two-dimensional wavelike disturbance propagates with a velocity c' which satisfies the inequality $0 < c' < V$, the linearized form of (1.1) may yield a singularity at the 'critical layer' where the liquid velocity equals c'. Such disturbances are outside the scope of the present analysis, but have been discussed previously by Lees & Lin (1946) and are also the subject of a future paper by Craik. Here, as for uniform films, it is found that the velocity c' of surface waves satisfies the condition $c' \geq V$, and the question of a singularity does not arise for these waves. However, the possibility of other wave modes with $0 < c' < V$ is not discounted: if, for example, the viscosity distribution possesses a near-discontinuity at some depth, 'internal' instabilities similar to those discussed by Yih (1967) may arise.

The assumption of constant density is also an approximation if temperature varies within the liquid. However, in the context of the present work, this is not a serious restriction. The major role of gravity is that due to the large density discontinuity at the liquid surface; and, provided the liquid remains stably stratified, comparatively small changes in density within the liquid are unlikely to be important for the waves under discussion.

Semenov (1964) has considered the stability of a horizontal flow in which viscosity increases exponentially with depth below the mean liquid surface, and where the motion is due to a constant tangential stress exerted at the liquid surface. He regards the viscosity as remaining constant in horizontal planes, even when the flow is given a small perturbation. In contrast, through (1.1), the present work takes into account the fluctuations of viscosity at a given location, which are due to the perturbed motion. Further, Semenov's model is not suitable for an examination of wind-generated waves, since it does not adequately incorporate the periodic stresses at the air–liquid interface which arise due to interaction of the air flow and the perturbed liquid surface. Such stress perturbations are included in the present analysis in the same manner as was employed by Craik (1966, 1968): the stresses are represented by suitable parameters and appropriate estimates for these may be substituted if desired.

Drazin (1962) has discussed the stability of parallel flows with variable viscosity and density, mainly at large Reynolds numbers. Some aspects of this work —particularly the initial formulation of the stability problem—are relevant to the present investigation, and these are mentioned later.

The manner in which viscosity varies with depth in the unperturbed flow need not be precisely specified. It is only necessary that the viscosity becomes sufficiently large with depth to ensure that most of the motion takes place in a fairly thin layer near the surface. The problem concerning wind-generated waves on a horizontal surface is solved for an arbitrary distribution of viscosity; and a convenient expression for the 'effective depth' h is found, which makes the stability criterion identical to that for a uniform film of thickness h. Also, for inclined flow under gravity, the analysis is pursued to a stage at which elementary numerical techniques may be employed to yield results for any specified viscosity distribution. Such results are presented for the particular case in which viscosity increases exponentially with depth.

The analyses may easily be extended to include the effects of surface contamination, as was done for uniform films by Benjamin (1963) and Craik (1968). However, this extension is quite straightforward, and is not worth including here. A further obvious extension is to cases in which the gravitational body force is replaced by an accelerating or decelerating frame of reference.

2. The basic equations

It is convenient to introduce dimensionless variables defined relative to the velocity V of the liquid surface, the constant density ρ of the liquid, and the length-scale h which is as yet unspecified, but which is a measure of the 'effective thickness' of the liquid layer. Dimensionless co-ordinates (x, y) are chosen such that the primary flow is in the x-direction, and y denotes the depth below the undisturbed free surface. The Reynolds number R equals $\rho Vh/\mu_0$ where μ_0 is the (constant) viscosity of particles comprising the liquid surface, and a 'dimensionless viscosity' m is defined as

$$m = \mu(x, y, t)/\mu_0.$$

A sketch of this configuration is shown in figure 1.

We denote the dimensionless pressure by p, and the dimensionless components of velocity and body force in the (x, y)-directions by (u, v) and $(G\sin\theta, G\cos\theta)$ respectively. Here, $G = gh/V^2$, where g is gravitational acceleration, and θ is the angle of inclination of the primary flow to the horizontal.

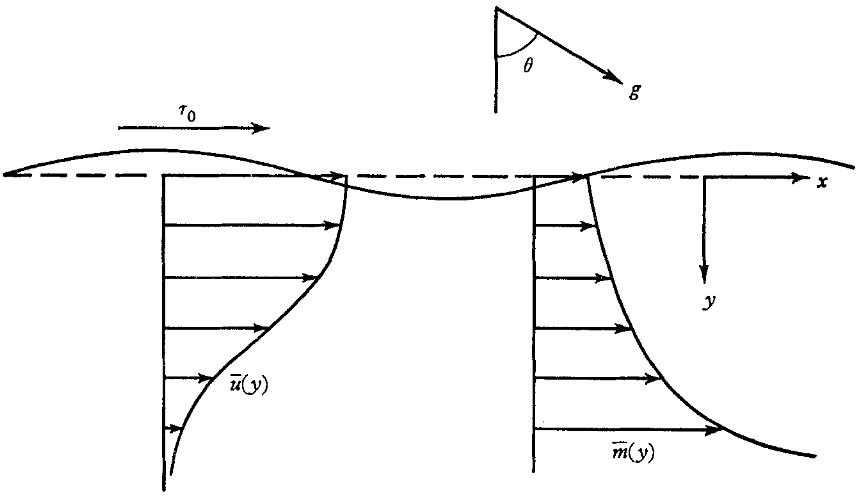

FIGURE 1. Sketch of flow configuration.

In the absence of any pressure gradient in the direction of motion, the primary flow is specified by

$$u = \bar{u}(y), \quad v = 0, \quad m = \bar{m}(y), \quad p = \bar{p}(y),$$

where

$$(\bar{m}\bar{u}')' = -GR\sin\theta, \quad \bar{p}' = G\cos\theta, \qquad (2.1\,a, b)$$

and the prime denotes differentiation with respect to y. The viscosity distribution $\bar{m}(y)$ may be regarded as a given property of the liquid, which satisfies the conditions

$$\bar{m}(0) = 1, \quad \bar{m}(\infty) = \infty;$$

and the corresponding velocity profile $\bar{u}(y)$ may be derived from equation (2.1a) and the boundary conditions

$$\bar{u}(0) = 1, \quad \bar{u}(\infty) = 0.$$

Also, if the liquid surface experiences a mean dimensional shear stress τ_0, we require that

$$\bar{u}'(0) = -R(\tau_0/\rho V^2), \qquad (2.2)$$

and this additional boundary condition yields an expression for the surface velocity V in terms of the other flow quantities.

3. The stability problem

We now consider that the primary flow experiences a small two-dimensional disturbance which is periodic in the x-direction. (The extension of the analysis to three-dimensional periodic disturbances may be effected by Squire's transformation: see Yih 1955). The normal displacement of the liquid surface is represented by

$$y = \eta(x,t) = \delta\, e^{i\alpha(x-ct)},$$

where α is the (real) dimensionless wave-number and c the dimensionless wave velocity, which may be complex. The wave amplitude is assumed to be sufficiently small for the problem to be linearized.

Continuity considerations permit the introduction of a perturbation stream function

$$\psi(x,y,t) = f(y)\,\eta(x,t),$$

such that the velocity components are

$$u = \bar{u}(y) + f'(y)\,\eta, \tag{3.1a}$$

$$v = -i\alpha f(y)\,\eta. \tag{3.1b}$$

Also, the dimensionless pressure and viscosity are of the form

$$p = \bar{p}(y) + \hat{p}(y)\,\eta,$$

$$m = \bar{m}(y) + \hat{m}(y)\,\eta.$$

The Navier–Stokes equations (with viscosity variation) together with (1.1) then yield the linearized results (cf. Drazin 1962)

$$\hat{p} = \bar{u}'f - (\bar{u}-c)f' + (i\alpha R)^{-1}\{-2\alpha^2\bar{m}f' + [\bar{m}(f''-\alpha^2 f) + \hat{m}\bar{u}']'\}, \tag{3.2a}$$

$$i\alpha R[(\bar{u}-c)(f''-\alpha^2 f) - \bar{u}''f] = \bar{m}(f^{\mathrm{iv}} - 2\alpha^2 f'' + \alpha^4 f) + 2\bar{m}'(f''' - \alpha^2 f')$$
$$+ \bar{m}''(f'' + \alpha^2 f) + \hat{m}(\bar{u}''' + \alpha^2\bar{u}') + 2\hat{m}'\bar{u}'' + \hat{m}''\bar{u}', \tag{3.2b}$$

$$(\bar{u}-c)\hat{m} = \bar{m}'f. \tag{3.2c}$$

We note that, when $\bar{m}(y) = 1$, the above equations are those for a homogeneous liquid. In particular, (3.2*b*) reduces to the Orr–Sommerfeld equation.

The boundary conditions at $y = \infty$ are

$$f(\infty) = f'(\infty) = 0. \tag{3.3a, b}$$

(If the flow is bounded by a rigid plane at some finite depth H, it is only necessary to let $\bar{m}(y)$ become infinite for all $y \geqslant H$.) At the liquid surface, there are three boundary conditions: two of these derive from the normal and tangential stress conditions at the surface, and the third is the linearized kinematic condition relating v and the surface displacement η. This last is

$$v = D\eta/Dt \quad (y = 0),$$

or

$$f(0) = c - 1. \tag{3.4}$$

The dimensionless normal and tangential stress perturbations exerted by an air stream on the perturbed liquid surface are of the form

$$\sigma_{yy} = -\Pi\eta, \quad \sigma_{xy} = \Sigma\eta, \tag{3.5a, b}$$

where Π and Σ are generally complex. The quantities Π and Σ depend on the properties of the air flow and the surface disturbance; but, for present purposes, it is convenient to regard them as complex parameters, the values of which may be estimated in particular cases (cf. Craik 1966, where the parameters Π and Σ

are the same as those defined above. The additional minus sign in (3.5a) arises since the direction of the y-axis is opposite to that employed by Craik.)

The requirement that capillary pressure and the normal stresses on either side of the liquid surface should be in equilibrium yields the linearized boundary condition

$$-\hat{p}(0)-2i\alpha R^{-1}f'(0) = (G\cos\theta-\Pi+\alpha^2T), \tag{3.6}$$

where $\hat{p}(0)$ is given by result (3.2a), where $T = \gamma(\rho h V^2)^{-1}$, and γ is the coefficient of surface tension at the liquid surface.

To linearized approximation, the tangential-stress perturbation just inside the liquid surface is found to be

$$\tau_i = R^{-1}[(\overline{m}\overline{u}')'+\hat{m}\overline{u}'+\overline{m}(f''+\alpha^2 f)]\eta \quad (y=0);$$

and the condition that this equals the tangential-stress perturbation σ_{xy} exerted by the air flow leads to the boundary condition

$$f''(0)+[\{\overline{u}''(0)-R\Sigma\}(c-1)^{-1}+\alpha^2]f(0) = 0, \tag{3.7}$$

on using results (3.2c), (3.4) and (3.5b).

The linearized characteristic-value problem is now completely specified by the equations (3.2a, b, c) together with the five boundary conditions (3.3a, b), (3.4), (3.6) and (3.7). Its solution yields an eigenvalue equation for the complex wave velocity c in terms of α and the other parameters of the problem.

4. The long-wave approximation

At this stage, it is convenient to introduce some approximations which are similar to those made in the analogous work on uniform films. We assume that

$$\alpha^2 \ll 1, \quad \alpha R, \alpha R|c| \ll 1 \tag{4.1 a, b}$$

(though, in §6, it is suggested that the conditions (4.1b) may be relaxed somewhat). These conditions require that the wavelength of the disturbance is large compared with the 'effective depth' h, and that, in the equations of motion, the inertia terms are small compared with the viscous terms. A first approximation to (3.2b) is then

$$(\overline{m}f''+\overline{u}'\hat{m})'' = 0.$$

On integrating twice and using result (3.2c), this becomes

$$\overline{m}f''+\frac{\overline{u}'\overline{m}'}{(\overline{u}-c)}f = Ay+B, \tag{4.2}$$

where A and B are constants of integration which may be determined from the boundary conditions. The corresponding approximations to the boundary conditions (3.6) and (3.7) are

$$\left(\overline{m}f''+\frac{\overline{u}'\overline{m}'}{\overline{u}-c}f\right)' = -i\alpha R(G\cos\theta-\Pi+\alpha^2T) \quad (y=0),$$

$$f'' = R\Sigma-\overline{u}'' \quad (y=0),$$

where results (3.2 a) and (3.4) have been used. Note that, although α^2 and αR are small, the term on the right-hand side of the former boundary condition must be retained, since the parameters G, Π and T may be large. It follows from these boundary conditions that the constants A and B are

$$A = -i\alpha R(G\cos\theta - \Pi + \alpha^2 T), \tag{4.3a}$$

$$B = R\Sigma - [(\overline{m}\overline{u}')']_{y=0} = R\Sigma + RG\sin\theta. \tag{4.3b}$$

With these values, it remains to solve (4.2) subject to the boundary conditions (3.3 a, b) and (3.4). This is done for two problems: namely, the onset of wind-generated waves on horizontal flows and the stability of inclined flows under gravity. The solution to the former problem yields the stability criterion to good approximation; but, for the latter, a better approximation is required, which incorporates the highest-order inertia terms from the equations of motion.

An equation similar to (4.2), but with $A = B = 0$, is discussed by Drazin. Two linearly independent solutions of this equation may be expressed as series of multiple integrals; but these are not required in the present work. Drazin also noted that the equation has a simple solution when $\overline{m}\overline{u}'$ is constant; and this case is examined in the next section.

5. Wind-generated waves on a horizontal flow

If the primary flow is horizontal, $\theta = 0$ and the motion is entirely due to the mean tangential stress τ_0 exerted by the air flow at the liquid surface. Then, from results (2.1) and (2.2),

$$\overline{m}\overline{u}' = -R(\tau_0/\rho V^2) \equiv D, \tag{5.1}$$

where D is a constant. For this case

$$\frac{\overline{m}'\overline{u}'}{\overline{m}(\overline{u}-c)} = -\frac{\overline{u}''}{\overline{u}-c},$$

and equation (4.2) becomes

$$(\overline{u}-c)f'' - \overline{u}''f = \frac{\overline{u}'(\overline{u}-c)}{D}(Ay+B) \equiv F(y).$$

First and second integrals of this equation, which satisfy the boundary conditions (3.3 a, b), are

$$(\overline{u}-c)f' - \overline{u}'f = -\int_y^\infty F(y_1)\,dy_1,$$

$$f = (\overline{u}-c)\int_y^\infty (\overline{u}-c)^{-2}dy_1\int_{y_1}^\infty F(y_2)\,dy_2.$$

The latter expression, together with the remaining boundary condition (3.4), yields

$$\int_0^\infty (\overline{u}-c)^{-2}dy\int_y^\infty F(y_1)\,dy_1 = -1,$$

and integrations by parts lead to the result

$$AI_1 + BI_2 = 2D, \tag{5.2}$$

where
$$I_1(c) = \int_0^\infty (\overline{u}-c)^{-2}\left\{\int_y^\infty [(\overline{u}-c)^2 - c^2]\,dy_1 - y[(\overline{u}-c)^2 - c^2]\right\}dy,$$

$$I_2(c) = \int_0^\infty (\overline{u}-c)^{-2}[(\overline{u}-c)^2 - c^2]\,dy.$$

Since A, B and D are known, and $\overline{u}(y)$ may be found for each prescribed viscosity distribution $\overline{m}(y)$, the complex wave velocity c may be determined from this equation. From now on, it will be convenient to denote the real and imaginary parts of complex quantities by the subscripts r and i.

It is of primary interest to determine the conditions for neutral stability, when c is real. However, if c is real and less than unity, the integrals I_1 and I_2 are singular at $y = y_c$, where $\overline{u}(y_c) = c$. As mentioned in the introduction, this possibility is not dealt with here: instead attention is restricted to those disturbances with $c_r > 1$, for which I_1 and I_2 are real when $c_i = 0$.

When $c_i = 0$, the real and imaginary components of (5.2) become

$$\left.\begin{aligned} A_r I_1(c_r) + B_r I_2(c_r) &= 2D, \\ A_i I_1(c_r) + B_i I_2(c_r) &= 0. \end{aligned}\right\} \qquad (5.3a, b)$$

Also, (4.3a, b) reveal that

$$\left.\begin{aligned} A_r &= -\alpha R \Pi_i, & A_i &= -\alpha R(G - \Pi_r + \alpha^2 T), \\ B_r &= R\Sigma_r, & B_i &= R\Sigma_i, \end{aligned}\right\} \qquad (5.4)$$

where the subscripts denote real and imaginary parts.

Particular estimates for Π and Σ, which derive from earlier work of Benjamin (1959), are given by Craik (1966), where their range of validity is also discussed. Here, we need only mention that $R|\Sigma|$ and $\alpha R|\Pi_i|$ are typically small compared with unity, that Π_i is negative and that Π_r, Σ_r and Σ_i are positive. In the following it is therefore permissible to assume that

$$R|\Sigma|, \quad \alpha R|\Pi_i| \ll 1, \qquad (5.5a, b)$$

as was done by Craik (1966). These assumptions again enable a simple solution to be found.

With these assumptions, $|A_r|$ and $|B_r|$ are small compared with unity; whereas, with an appropriate choice of length scale h, the value of $|D|$ is $O(1)$. Therefore, if (5.3a) is to be satisfied, either I_1 or I_2 must be large compared with unity. However, for disturbances with $c_r > 1$, it is easily verified that the magnitudes of I_1 and I_2 are $O(1)$ except when c_r is close to unity. (It should be recalled that, with an appropriate length-scale h, $\overline{u}$ is very small for depths y greater than $O(1)$, due to the large values of the viscosity there: the main contributions to the integrals I_1 and I_2 then derive from a layer near the surface whose dimensionless depth is $O(1)$.) If $c_r = 1 + \epsilon$, say, where ϵ is a small *positive* quantity, the integrands become very large near $y = 0$, and their contributions to I_1 and I_2 are correspondingly increased. Therefore, in order that I_1 or I_2 may be sufficiently large to satisfy equation (5.3a), it is necessary that $c_r \simeq 1+$. This is similar to the corresponding result for uniform thin films (see Craik 1966, equation (7.2)) that the phase velocity of infinitesimal waves is nearly equal to the velocity of the liquid surface.

This result enables simplification of (5.3b). On writing $c = c_r = 1 + \epsilon$ in I_1 and I_2, and examining the contributions to these integrals from the vicinity of $y = 0$, it is easily verified that

$$I_1(1+\epsilon) = \frac{1}{\overline{u}'(0)\epsilon}\int_0^\infty \overline{u}(2-\overline{u})\,dy + O(\log\epsilon), \quad I_2(1+\epsilon) = \frac{1}{\overline{u}'(0)\epsilon} + O(1).$$

On retaining only the highest-order terms in ϵ, equations (5.3*b*) and (5.4) yield the result

$$(G-\Pi_r+\alpha^2T)\int_0^\infty \bar{u}(2-\bar{u})\,dy = \Sigma_i/\alpha,$$

which represents, to good approximation, the condition for neutral stability. In terms of dimensional quantities, it becomes

$$(\rho g-P_r+k^2\gamma)\int_0^\infty U(2V-U)\,dy' = (kh)^{-1}V^2hT_i, \tag{5.6}$$

where P_r and T_i are dimensional stress parameters defined as

$$P_r = (\rho V^2h^{-1})\,\Pi_r, \quad T_i = (\rho V^2h^{-1})\,\Sigma_i,$$

and $U(y')$ denotes the dimensional primary velocity profile as a function of the actual depth y'.

The corresponding result for a uniform film of thickness h is (Craik 1966, equation (7.1*b*))

$$\rho g-P_r+k^2\gamma = \tfrac{3}{2}(kh)^{-1}T_i, \tag{5.7}$$

and this may be recovered from (5.6) by setting

$$U = V(1-y'/h) \quad (0\leqslant y'\leqslant h), \qquad U=0 \quad (y'>h).$$

The similarity between results (5.6) and (5.7) permits a very convenient definition of the 'effective depth' h of liquids with viscosity stratification (up till now, h has not been precisely defined). For, if h is taken to be

$$h = \tfrac{3}{2}V^{-2}\int_0^\infty U(2V-U)\,dy', \tag{5.8}$$

these equations become identical. Then, the criterion for neutral stability with viscosity stratification is *precisely the same* as that for uniform films.

When c_i is non-zero, but is sufficiently small that

$$|c_i| \ll |c_r-1|, \quad 1,$$

an analysis similar to that above yields the approximate result

$$\alpha Rc_i = \tfrac{1}{2}(\alpha R)^2\left[(\Sigma_i/\alpha)-(G-\Pi_r+\alpha^2T)\int_0^\infty(2-\bar{u})\,\bar{u}\,dy\right].$$

With the above choice of h, this equation is also identical to the corresponding result (Craik 1966, equation (7.3)) for uniform films.

It is clear that all the results obtained by Craik (1966) for uniform films are directly applicable to the present situation. Accordingly, further details need not be presented here: instead, the reader is referred to §§7–10 of Craik's paper, which examine the neutral case, the stability curves and the range of validity of the approximations. Also, following Craik (1968), the above analysis may easily be extended to include the effects of contamination by insoluble surface-active agents, which are often significant in practice.

Finally, it should be recalled that the stability criterion examined here refers only to 'surface-wave' modes. As mentioned in the introduction, there remains the possibility of unstable 'internal-wave' modes with $0 < c_r < 1$, similar to those examined by Yih (1967) for two superposed liquid layers.

6. Inclined flow under gravity

We now consider cases where the primary flow is entirely due to the body force component $G\sin\theta$ in the direction of motion. Since the air flow is absent, the mean shear stress τ_0 and the stress perturbations σ_{yy} and σ_{xy} are zero. From results (2.1*a*) and (2.2),

$$\overline{m}\overline{u}' = -GR\sin\theta\, y,$$

and

$$\frac{\overline{m}'\overline{u}'}{\overline{m}(\overline{u}-c)} = \frac{(\overline{u}'/y)-\overline{u}''}{\overline{u}-c}.$$

Equation (4.2) is therefore

$$f''+\frac{(\overline{u}'/y)-\overline{u}''}{\overline{u}-c}f = \frac{-(\overline{u}'/y)}{GR\sin\theta}(Ay+B), \tag{6.1}$$

and the appropriate values of A and B are, from (4.3*a*, *b*),

$$A = -i\alpha R(G\cos\theta+\alpha^2T), \quad B = RG\sin\theta.$$

We now require the solution $f(y)$ and the wave velocity c which satisfy equation (6.1) and the boundary conditions (3.3*a*, *b*) and (3.4), for a prescribed velocity profile $\overline{u}(y)$.

For the present problem, the depth scale h is not yet precisely defined. Without loss of generality, it may be chosen such that

$$RG\sin\theta = 1. \tag{6.2}$$

It follows that B may be taken as unity in (6.1); this step being equivalent to defining the depth scale h to be $(\mu_0 V/\rho g\sin\theta)^{\frac{1}{2}}$.

When α is sufficiently small, $|A|$ is small compared with $|B|$, and a first approximation to (6.1)—which is itself an approximation to (3.2*b*)—may be obtained on setting A equal to zero. Denoting the approximate solution by the subscript zero, we require to find $f_0(y)$ and c_0 which satisfy the equation

$$f_0''+\left\{\frac{\overline{u}'/y-\overline{u}''}{\overline{u}-c_0}\right\}f_0 = -(\overline{u}'/y) \tag{6.3}$$

and the boundary conditions

$$f_0(\infty) = f_0'(\infty) = 0, \quad f_0(0) = c_0-1.$$

The associated viscosity variation $\hat{m}_0(y)$ is obtained from (3.2*c*) as

$$\hat{m}_0(y) = \frac{\overline{m}'f_0}{\overline{u}-c_0} = \frac{(\overline{u}''y-\overline{u}')}{\overline{u}'^2(\overline{u}-c_0)}f_0. \tag{6.4}$$

For prescribed values of $\overline{u}(y)$, the appropriate solutions for $f_0(y)$ and c_0 may be obtained numerically, using an iteration procedure. To this approximation, c_0 is real and the solution represents a neutrally stable wave.

In order to derive the stability criterion, second order approximations for $f(y)$ and c are required. For this, the term in A and the highest-order inertia terms must be included in the analysis. The inclusion of the latter requires that we return to (3.2). Following Yih (1963), we write

$$\left.\begin{aligned} f(y) &= f_0(y) + i\alpha R f_1(y), \\ \hat{m}(y) &= \hat{m}_0(y) + i\alpha R \hat{m}_1(y), \\ c &= c_0 + i\alpha R c_1. \end{aligned}\right\} \tag{6.5}$$

These expressions may be regarded as the leading terms of a power-series expansion in terms of the small parameters αR and α^2, when α^2 is also small compared with αR. Retaining only those terms of (3.2*b*) which are $O(\alpha R)$, we find that

$$(\overline{m} f_1'' + \overline{u}' \hat{m}_1)'' = (\overline{u} - c_0) f_0'' - \overline{u}'' f_0, \tag{6.6}$$

$$\hat{m}_1 = \frac{\overline{m}' f_1}{\overline{u} - c_0} + \frac{\overline{m}' f_0 c_1}{(\overline{u} - c_0)^2}. \tag{6.7}$$

First and second integrals of (6.6) are

$$(\overline{m} f_1'' + \overline{u}' \hat{m}_1)' = (\overline{u} - c_0) f_0' - \overline{u}' f_0 + C, \tag{6.8}$$

$$\overline{m} f_1'' + \overline{u}' \hat{m}_1 = \int_0^y [(\overline{u} - c_0) f_0' - \overline{u}' f_0] \, dy_1 + Cy + D, \tag{6.9}$$

where C and D are disposable constants. To the same order of approximation, the boundary conditions (3.6), (3.7) and (3.4) yield

$$(\overline{m} f_1'' + \overline{u}' \hat{m}_1)' = (1 - c_0) f_0' + G \cos\theta + \alpha^2 T \quad (y = 0),$$

$$\overline{m} f_1'' + \overline{u}' \hat{m}_1 = 0 \quad (y = 0),$$

$$f_1 = c_1 \quad (y = 0).$$

The first two of these boundary conditions, together with (6.8) and (6.9), determine C and D to be

$$C = G \cos\theta + \alpha^2 T, \quad D = 0.$$

On substituting in (6.9) for C, D, $\overline{m}$ and $\hat{m}_1$ we obtain

$$f_1'' + \left\{\frac{(\overline{u}'/y) - \overline{u}''}{\overline{u} - c_0}\right\} f_1 = -c_1 \left\{\frac{(\overline{u}'/y) - \overline{u}''}{(\overline{u} - c_0)^2}\right\} f_0 - (\overline{u}'/y) I - \overline{u}'(G\cos\theta + \alpha^2 T), \tag{6.10}$$

where

$$I \equiv \int_0^y [(\overline{u} - c_0) f_0' - \overline{u}' f_0] \, dy_1 = (\overline{u} - c_0) f_0 + (1 - c_0)^2 - 2 \int_0^y \overline{u}' f_0 \, dy_1, \tag{6.11}$$

on integration by parts. Also, from (6.3),

$$(\overline{u} - c_0) y f_0'' + (\overline{u}' - y \overline{u}'') f_0 = -\overline{u}'(\overline{u} - c_0);$$

and this equation may be integrated from 0 to y, to find, after integration by parts, that

$$3 \int_0^y \overline{u}' f_0 \, dy_1 = (\overline{u} - c_0 + \overline{u}' y) f_0 - (\overline{u} - c_0) y f_0' - \tfrac{1}{2}(\overline{u} - c_0)^2 + \tfrac{3}{2}(1 - c_0)^2. \tag{6.12}$$

On using results (6.11) and (6.12), equation (6.10) becomes

$$f_1'' + \left\{\frac{(\overline{u}'/y) - \overline{u}''}{\overline{u} - c_0}\right\} f_1 = -c_1 \left\{\frac{(\overline{u}'/y) - \overline{u}''}{(\overline{u} - c_0)^2}\right\} f_0 - \overline{u}'(G\cos\theta + \alpha^2 T) + H(y), \tag{6.13}$$

where $\quad H(y) \equiv -\tfrac{1}{3}(\overline{u}'/y)[(\overline{u} - c_0 - 2\overline{u}' y) f_0 + 2(\overline{u} - c_0) y f_0' + (\overline{u} - c_0)^2].$

In addition, $f_1(y)$ and c_1 must be such that the boundary conditions

$$f_1(\infty) = f_1'(\infty) = 0, \quad f_1(0) = c_1 \tag{6.14}$$

are satisfied.

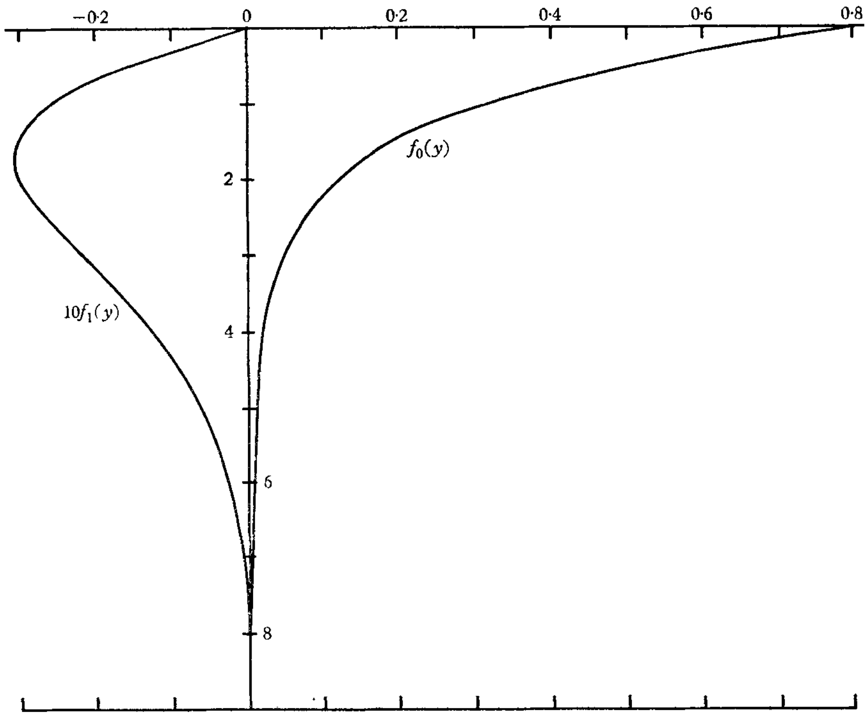

FIGURE 2. The functions $f_0(y)$ and $10f_1(y)$ for the viscosity distribution $m(y) = e^y$.

With a given dimensionless velocity profile $\bar{u}(y)$, the first approximations c_0 and $f_0(y)$ may be calculated numerically from (6.3), as mentioned above. When these are known, a similar iterative procedure yields solutions $c_1, f_1(y)$ to the above problem, which correspond to chosen values of the parameter

$$(G\cos\theta + \alpha^2 T).$$

It is clear that c_1 must be real; and, since the imaginary part of the wave velocity c is $i\alpha Rc_1$, the sign of c_1 determines whether the wave is stable or unstable.

Such calculations have been carried out for the particular velocity profile

$$\bar{u}(y) = (1+y)\,e^{-y}, \tag{6.15}$$

which is found, from the work of §2, to occur in the interesting case where the viscosity increases exponentially with depth. The corresponding dimensional and dimensionless viscosity distributions are

$$\mu(y') = \mu_0 e^{y'/h}, \quad \bar{m}(y) = e^y \quad (y, y' \geqslant 0).$$

(It may be verified that the constant h used here is such that condition (6.2) is satisfied.) The value of c_0 was found to be

$$c_0 = 1{\cdot}81,$$

which represents a neutral wave travelling with a velocity somewhat less than twice that of the liquid surface. This contrasts with the result of Benjamin and Yih for a uniform film, that the wave velocity is just twice that of the liquid surface. The solution $f_0(y)$ is shown in figure 2.

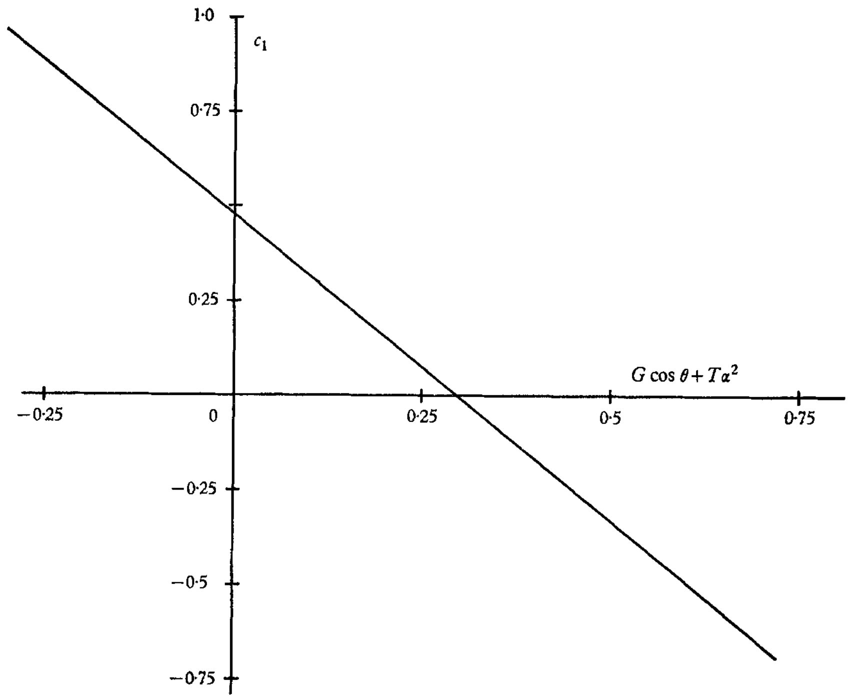

FIGURE 3. Curve of $c_1 (= c_i/\alpha R)$ against $(G\cos\theta + T\alpha^2)$ for the viscosity distribution $\overline{m}(y) = e^y$.

In the next approximation, values of c_1 were found which correspond to several chosen values of $(G\cos\theta + \alpha^2 T)$. These results are shown in figure 3. The curve of c_1 against $(G\cos\theta + \alpha^2 T)$ resembles a straight line, but no satisfactory explanation for this has been found. A disturbance of wave-number α is stable or unstable according as $(G\cos\theta + \alpha^2 T)$ is greater or less than 0·295. The function $f_1(y)$ corresponding to the neutral case $c_1 = 0$ is shown in figure 2. In the range $0 \leqslant y < 2$, where most of the motion takes place, $|f_1(y)|$ is considerably smaller than $|f_0(y)|$, and the same is true of their first derivatives. This fact suggests that condition (4.1*b*) may be unnecessarily severe, and that the present approximate theory may hold for all values of αR less than $O(1)$.

When
$$G\cos\theta < 0{\cdot}295,$$
very long waves ($\alpha \to 0$) are unstable. Also, on using result (6.2), this instability condition becomes
$$R > 3{\cdot}39\cot\theta.$$
Now, for the velocity profile (6.15), the volumetric flow rate per unit span Q is
$$Q \equiv Vh\int_0^\infty \overline{u}(y)\,dy = 2Vh;$$

therefore, in terms of Q, the instability condition is

$$(\rho/\mu_0)Q > 6{\cdot}78\cot\theta. \tag{6.16}$$

This may be compared with the result

$$(\rho/\mu)Q > \tfrac{5}{6}\cot\theta,$$

which was found for uniform films by Benjamin and Yih. Once again, instability is predicted whenever the liquid surface is vertical, for $\cot\theta$ is then zero. For $\theta < 90°$, the flow is stable provided Q is sufficiently small, but instability occurs when Q is large enough to satisfy condition (6.16).

The above analysis may readily be extended to include the effects of surface contamination, by following the method of Benjamin (1963). It is clear that the instability examined above is very similar to that occurring in uniform films, and further discussion is therefore unnecessary.

Part of this work was performed while one of us (F.S.) was the recipient of an S.R.C. research studentship. The computations were carried out on the IBM 1620 computer of the University of St Andrews.

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